Indian Statistical Institute, Bangalore
B. Math (II), First semester 2017-2018
End-Semester Examination : Statistics (I)

Date: 20-11-2017 **Maximum Score 60** **Duration: 3 Hours**

1. Define ρ_{XY} the correlation coefficient between X and Y.

   Let F be distribution function of a continuous random variable. Let $X_1,\ X_2,\cdots,X_n$ be *a random sample* from F. Consider real numbers $a_0, a_1, a_2, \cdots, a_{k-1}, a_k,\ k>1$, such that $-\infty = a_0 < a_1 < \cdots < a_{k-1} < a_k = \infty$. Define $Y_j = \#\{i : X_i \in (a_{j-1}, a_j]\}\,;\ 1 \le j \le k-1$ and $Y_k = \#\{i : X_i \in (a_{k-1}, a_k)\}$. Find $\rho_{Y_iY_j}$, the correlation coefficient between Y_i and Y_j; $1 \le i < j \le k$.

   $[2+8=10]$

2. In the regression framework $y_i = \beta x_i + \varepsilon_i,\ 1 \le i \le n$, given $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ the errors ε_i are *iid* $N(0,\sigma^2)$. Obtain *least squares estimator for* β. Obtain an unbiased estimator $\widehat{\sigma^2}_{\texttt{LSE}}$ for σ^2 based on *sum of squares of residuals.* Also obtain $\widehat{\sigma^2}_{\texttt{MLE}}$, *maximum likelihood estimator (MLE)* for σ^2. Does $\widehat{\sigma^2}_{\texttt{MLE}}$ agree with $\widehat{\sigma^2}_{\texttt{LSE}}$?

   $[5+4+6+1=16]$

3. Let $X_1,\ X_2,\cdots,X_n$ be *a random sample* from $N(0,\sigma^2)$. Consider the hypothesis testing problem

   $$H_0\colon\ \sigma^2 = \sigma_0^2 \text{ versus } H_1\colon\ \sigma^2 \ne \sigma_0^2.$$

   Let $Y = a\sum_{i=1}^{m} X_i + b\sum_{i=m+1}^{n} X_i$, where a, b are real numbers, at least one of them being nonzero. Obtain a *statistic* W as a function of Y that can work as an unbiased estimator for σ^2. Obtain the distribution of W. Develop, explaining the heuristics, a test based on W for the above problem.

   $[3+5+6=14]$

4. Consider random variable X with *probability density function* (*pdf*) $f(x|\alpha,\lambda,\delta,k,\theta)$; where $\alpha>0, \lambda>0, \delta>0, k>0$, that is proportional to

   $$g(x|\alpha,\lambda,\delta,k,\theta) = \left(\frac{x-\theta}{\delta}\right)^{\alpha k-1} \exp\left[-\lambda\left(\frac{x-\theta}{\delta}\right)^{k}\right] I_{[\theta,\infty)}(x).$$

   (a) For $k=1$ obtain $E(X^2)$.

   (b) For $k=1$ and known δ,θ obtain *method of moments (MOM) estimators* for α and λ.

   (c) For $\alpha = 11$, and known λ,δ,k,θ how would you draw observations on X using a *Direct Method?*

   $[(2+6)+8+8=24]$